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Technical Note

1968-26

Practical Designs
for RC Active Filters
Using Operational Amplifiers

J. W. Craig

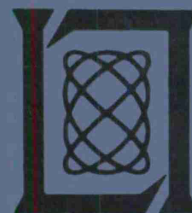
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USING OPERATIONAL AMPLIFIERS

J. W. CRAIG

Group 62

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ABSTRACT

Several practical circuits for the design of RC active filters using operational amplifiers are given. The transfer function of each circuit has a pair of complex poles and has transmission zeros either all at infinity, one each at zero and infinity or an imaginary pair at $\pm i \omega_0$. More complicated transfer functions can be obtained by cascading these circuits. Design procedures are outlined which minimize the need of capacitor trimming, and the sensitivity of the transfer function to changes in gain of the operational amplifier is minimized.

Accepted for the Air Force
Franklin C. Hudson
Chief, Lincoln Laboratory Office

PRACTICAL DESIGNS FOR RC ACTIVE FILTERS USING OPERATIONAL AMPLIFIERS

I. INTRODUCTION

The availability of high performance but inexpensive integrated circuit operational amplifiers makes them very attractive for use in the realization of RC active filters for frequencies below approximately 100 kHz. Elimination of the inductor is a great advantage in both size and weight, especially at very low frequencies. There are a great many ways of realizing RC active filters, so the circuits given in this report are only a few of these available. However, they are essentially a complete design catalog in the sense that all transfer functions with complex conjugate poles and with zeros anywhere on the $j\omega$ axis can be realized as a cascade of the basic circuits or sections.

All of the circuits are easy to design and adjust, and their performance is exactly as designed. In all cases most or all of the capacitors do not need to be trimmed to a predetermined value; the measured values of the capacitors are used to compute the values of the resistors which are then trimmed to these values. Assembly of the components results in a working circuit without further adjustment of the components.

All of the circuits presented in this report are special cases of a differential input operational amplifier with feedback provided by a 4-terminal, grounded, RC network. The general circuit arrangement is analyzed and the results are applied to each particular circuit. The emphasis is on the design of the individual circuits or building blocks, each of which realizes a pair of complex conjugate poles and 0, 1 or 2 zeros of a transfer function.

II. ANALYSIS OF AN OPERATIONAL AMPLIFIER WITH A GENERAL FEEDBACK NETWORK

The form of the circuit to be considered is shown in Fig. 1. The input to the network is the voltage source E_1 at terminal #1 and the output is voltage E_3 at terminal #3, the output of the operational amplifier. The operational amplifier is assumed to be ideal

in the sense that its input admittance is zero (I_2 and I_4 are zero) and its output impedance is zero. For the purpose of analysis the gain K of the amplifier is assumed

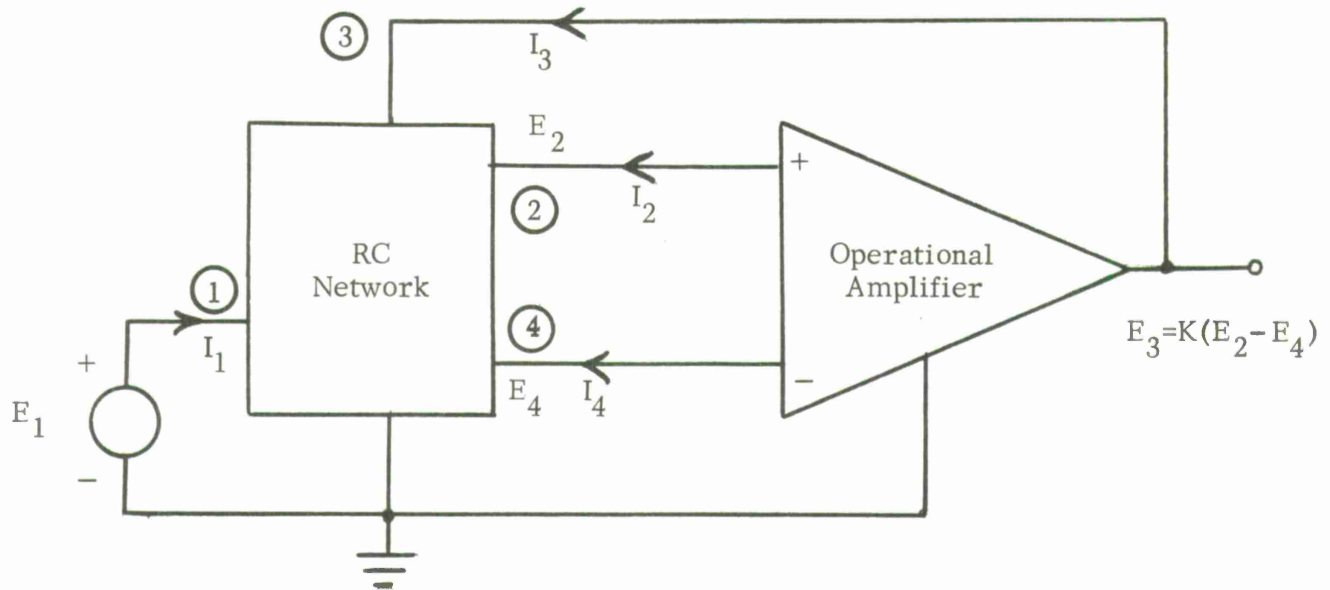


Fig. 1. The general form of the RC active section under considerations. Terminal numbers and corresponding voltages and currents are labeled.

finite so that the effect of large but not infinite gain can be determined.

The node equations of the RC network are:

$$I_1 = y_{11} E_1 + y_{12} E_2 + y_{13} E_3 + y_{14} E_4 \quad (1)$$

$$I_2 = y_{21} E_1 + y_{22} E_2 + y_{23} E_3 + y_{24} E_4 = 0 \quad (2)$$

$$I_3 = y_{31} E_1 + y_{32} E_2 + y_{33} E_3 + y_{34} E_4 \quad (3)$$

$$I_4 = y_{41} E_1 + y_{42} E_2 + y_{43} E_3 + y_{44} E_4 = 0 \quad (4)$$

In Eqs. (2) and (4), I_2 and I_4 have been made to equal to zero. Since the output voltage $E_3 = K(E_2 - E_4)$, solving for E_2 gives

$$E_2 = \frac{E_3}{K} + E_4$$

Substitute this for E_2 in Eqs. (2) and (4), and rearrange terms to obtain the two equations

$$-y_{21} E_1 = (y_{23} + \frac{y_{22}}{K}) E_3 + (y_{22} + y_{24}) E_4 \quad (5)$$

$$-y_{41} E_1 = (y_{43} + \frac{y_{42}}{K}) E_3 + (y_{44} + y_{42}) E_4 \quad (6)$$

Solving Eqs. (5) and (6) for E_3 yields for the transfer function $\frac{E_3}{E_1}$,

$$\frac{E_3}{E_1} = \frac{y_{41}(y_{22} + y_{24}) - y_{21}(y_{44} + y_{42})}{y_{23}(y_{44} + y_{42}) - y_{43}(y_{22} + y_{24}) + \frac{1}{K}(y_{22}y_{44} - y_{24}y_{42})} \quad (7)$$

An alternate form is

$$\frac{E_3}{E_1} = \frac{-K \left[\frac{y_{21}(y_{44} + y_{42}) - y_{41}(y_{22} + y_{24})}{y_{22}y_{44} - y_{24}y_{42}} \right]}{1 + K \left[\frac{y_{23}(y_{44} + y_{42}) - y_{43}(y_{22} + y_{24})}{y_{22}y_{44} - y_{24}y_{42}} \right]} \quad (8)$$

It can be shown that the expression within the brackets in the numerator is equal to what might be called the "feed forward" ratio of the RC network:

$$\left(\frac{E_4 - E_2}{E_1} \right)_{E_3 = 0} \quad (9)$$

and that the expression within the brackets in the denominator is equal to the feedback ratio of the RC network:

$$\left(\frac{E_4 - E_2}{E_3} \right)_{E_1 = 0} \quad (10)$$

The voltage ratio in (9) is evaluated with node 3 grounded, and the ratio in (10) is evaluated with node 1 grounded. This gives an interesting and satisfying physical interpretation of the transfer function as the negative of the ratio of forward gain, $K T_F$, to unity plus the loop gain, $K T_{FB}$:

$$T(s) = \frac{E_3}{E_1} = \frac{-K \left(\frac{E_4 - E_2}{E_1} \right) \bigg|_{E_3=0}}{1 + K \left(\frac{E_4 - E_2}{E_3} \right) \bigg|_{E_1=0}} = \frac{-K T_F}{1 + K T_{FB}} = \frac{-T_F/T_{FB}}{1 + (K T_{FB})^{-1}} \quad (11)$$

The sensitivity of the transfer function to changes in gain of the operational amplifier is a critical parameter of the circuits considered here, and its standard definition is the ratio of the fractional change in the transfer function to the fractional change in the amplifier gain, which for small changes is

$$S_T^{(K)} = \frac{\partial T(s)}{\partial K} \frac{K}{T}.$$

For the transfer function of Eq. (11) the sensitivity is

$$S_T^{(K)} = \frac{1}{1 + K T_{FB}} \quad (12)$$

We see from Eqs. (11) and (12) that if the loop gain, $K T_{FB}$, is much greater than unity, the transfer function is essentially independent of the amplifier gain and its sensitivity to gain changes is very small.

For an amplifier with very high gain, the transfer function approaches

$$T(s) = \frac{-T_F}{T_{FB}} = \frac{y_{41}(y_{22} + y_{24}) - y_{21}(y_{44} + y_{42})}{y_{23}(y_{44} + y_{42}) - y_{43}(y_{22} + y_{24})} \quad (13)$$

The form of the numerator and denominator show that with proper design, complex zeros and complex poles can be realized by this transfer function. Thus far the reciprocal

properties of the RC network (i. e. that $y_{ij} = y_{ji}$) have not been used explicitly and this fact does not greatly simplify the transfer function, Eq. (12). In fact even for a reciprocal network the full generality of this transfer function will not be used here. It is hoped that future work will be able to exploit Eq. (12) for the realization of more complex transfer functions than are developed in this report. For our present purpose the special case of the transfer function with $y_{24} = y_{42} = 0$ and $y_{14} = y_{41} = 0$ is sufficient. The resulting transfer function is

$$\frac{E_3}{E_1} = \frac{-y_{21} y_{44}}{y_{23} y_{44} - y_{43} y_{22}} \quad (14)$$

Using this transfer function we will develop a catalog of circuits and their design.

III. SIMPLE RC FEEDBACK NETWORK

The first network we will consider is shown in Fig. 2. Each admittance in the network is either a conductance or a capacitance. For this network all of the y_{ij} 's do not exist but it can be analyzed using Eq. (11). For this circuit T_F and T_{FB} are

$$T_F = \frac{Y_1 Y_4}{Y_4(Y_1 + Y_2 + Y_3) + Y_5(Y_1 + Y_2 + Y_3 + Y_4)} \quad (15)$$

and

$$T_{FB} = \frac{Y_3 Y_4 + Y_5(Y_1 + Y_2 + Y_3 + Y_4)}{Y_4(Y_1 + Y_2 + Y_3) + Y_5(Y_1 + Y_2 + Y_3 + Y_4)} \quad (16)$$

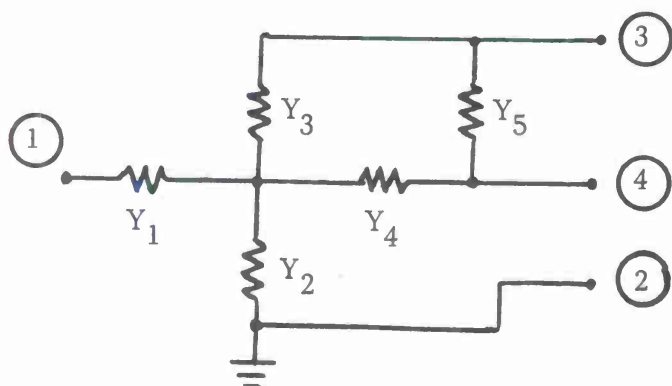


Fig. 2. Simple RC feedback network.

The situation of greatest interest is that of very large operational amplifier gain. The complete transfer function will be given for infinite gain, but later in the development the effect of large but finite gain will be given. For infinite gain the transfer function is

$$T(s) = \frac{E_3}{E_1} = - \frac{T_F}{T_{FB}} = \frac{-Y_1 Y_4}{Y_3 Y_4 + Y_5 (Y_1 + Y_2 + Y_3 + Y_4)} \quad (17)$$

There are two cases of interest: the low-pass case with all transmission zeros at infinity and the bandpass case with one transmission zero at the origin and the other at infinity. In both cases a pair of complex poles is required.

A. Low-Pass

For this case Y_2 and Y_5 must be capacitive and the other three admittances resistive. The complete circuit is shown in Fig. 3 and its transfer function is

$$\frac{E_3}{E_1} = \frac{-g_1 g_4 / C_2 C_5}{s^2 + s (g_1 + g_3 + g_4) / C_2 + (g_3 g_4 / C_2 C_5)} \quad (18)$$

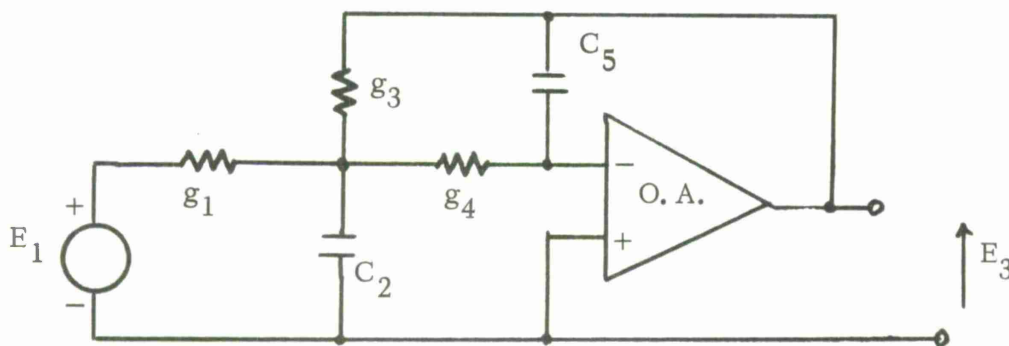


Fig. 3. Active lowpass filter.

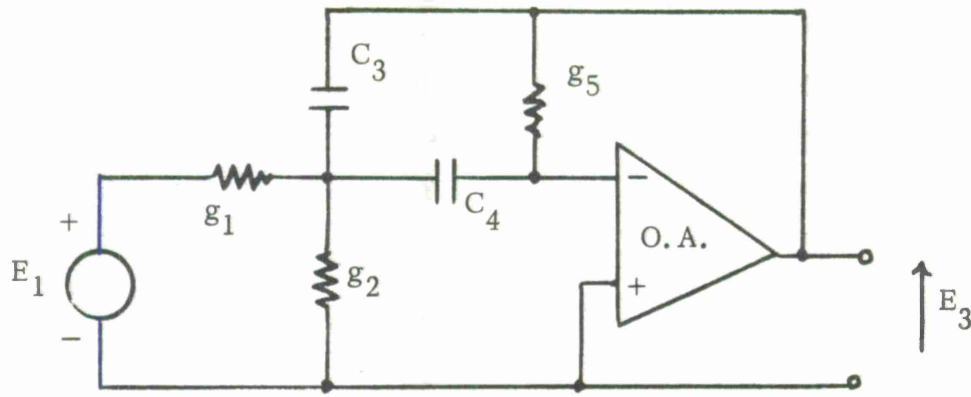


Fig. 4. Active bandpass filter.

which must be equal to the desired transfer function that has the form

$$T(s) = \frac{-K_o \omega_o^2}{s^2 + 2\alpha s + \omega_o^2} \quad , \quad (19)$$

where $-K_o$ is the DC gain, $T(o)$, and $T(s)$ has poles at $s = -\alpha \pm i\beta$ with $\omega_o^2 = \alpha^2 + \beta^2$.

By equating like coefficients in Eqs. (18) and (19) we obtain: $\omega_o^2 = g_3 g_4 / C_2 C_5$,

$2\alpha = (g_1 + g_3 + g_4) / C_2$ and $K_o = g_1 / g_3$. Choose the following parameters

$$\sigma_1 = \frac{g_4}{C_5} \quad , \quad \sigma_2 = \frac{g_3}{C_2} \quad \text{and} \quad \rho = \frac{C_2}{C_5} \quad .$$

Then we find that

$$2\alpha = \frac{\sigma_1}{\rho} + \sigma_2 (1 + K_o) \quad \text{and} \quad \omega_o^2 = \sigma_1 \sigma_2$$

so

$$\sigma_1^2 - 2\alpha \rho \sigma_1 + \omega_o^2 (1 + K_o) \rho = 0 \quad . \quad (20)$$

The solution for σ_1 is

$$\sigma_1 = \alpha \rho \pm \left[(\alpha \rho)^2 - \omega_o^2 (1+K_o) \rho \right]^{\frac{1}{2}} \quad (21)$$

For realizability σ_1 must be positive and real. so the expression under the radical in Eq. (21) must be positive which requires $\rho > 4Q^2(1+K_o) = \rho_{\min}$, where $Q = \omega_o/2\alpha$. Let

$$\rho = \frac{4Q^2(1+K_o)}{1-\gamma^2} = \frac{\rho_{\min}}{1-\gamma^2} \quad \text{where } \gamma \text{ is a positive parameter } (0 < \gamma < 1)$$

that determines how much greater ρ is than ρ_{\min} . Then in terms of ρ, Q, K_o and γ the solutions for σ_1 and σ_2 are

$$\frac{\sigma_1}{\omega_o} = \frac{g_4}{\omega_o C_5} = \frac{\rho}{2Q} (1 \pm \gamma) = \frac{2Q(1+K_o)}{1 \mp \gamma} \quad (22)$$

and

$$\frac{\sigma_2}{\omega_o} = \frac{g_3}{\rho \omega_o C_5} = \frac{1 \mp \gamma}{2Q(1+K_o)} \quad (23)$$

This last equation gives

$$\frac{g_3}{\omega_o C_5} = \frac{2Q}{1 \pm \gamma} \quad (24)$$

In these equations where there is a choice of sign, either the upper or lower sign can be selected, but the use must be consistent.

A convenient design approach is the following:

- 1) compute Q from the specified pole location.
- 2) choose K_o and compute ρ_{\min} .
- 3) select capacitors for C_2 and C_5 whose ratio C_2/C_5 is greater than ρ_{\min} , measure their values and compute γ .
- 4) compute g_4 and g_3 from Eqs. (22) and (24).

This scheme does not require the trimming of capacitors.

For this circuit T_{FB} of Eq.(16) has a minimum value of approximately

$\frac{1}{(1+K_o)Q^2}$ at $\omega \approx \omega_o$, so for $|K T_{FB}|$ to be $\gg 1$, K must be much greater than

$(1+K_o)Q^2$. The dependence of the minimum amplifier gain on the square of Q severely limits the highest Q that can be realized reliably with this circuit.

B. Bandpass

With Y_3 and Y_4 capacitive and the other three admittances resistive the transfer function has a bandpass character of the form

$$T(s) = \frac{-2\alpha s T_{\max}}{s^2 + 2\alpha s + \omega_o^2} = \frac{-s g_1 / C_3}{s^2 + s \left(\frac{1}{C_3} + \frac{1}{C_4} \right) g_5 + \frac{g_5(g_1 + g_2)}{C_3 C_4}} \quad (25)$$

The circuit is shown in Fig. 4. As before $\omega_o^2 = \alpha^2 + \beta^2$ and we will again use $Q = \omega_o / 2\alpha$. $T_{\max} = |T(i\omega_o)|$ is the maximum value of the transfer function and is a useful design parameter. For this circuit the capacitors C_3 and C_4 can have the same value which is a convenience. By equating corresponding coefficients in Eq.(25), the following three equations are obtained:

$$2\alpha = g_5 \left(\frac{1}{C_3} + \frac{1}{C_4} \right) \quad (26)$$

$$T_{\max} = \frac{g_1}{C_3} \frac{1}{g_5 \left(\frac{1}{C_3} + \frac{1}{C_4} \right)} = \frac{g_1}{2\alpha C_3} \quad (27)$$

$$\omega_o^2 = \frac{g_5(g_1 + g_2)}{C_3 C_4} = \frac{2\alpha(g_1 + g_2)}{C_3 + C_4} \quad (28)$$

As before the design is based on the assumption that a nominal value is selected for $C_3 = C_4$ and that measured values of the components to be used for C_3 and C_4 are available. Calculate g_5 and g_1 using Eqs. (26) and (27). Then compute g_2 from Eq. (28).

The design equations are, in order,

$$g_5 = \frac{1}{R_5} = \frac{2\alpha}{\frac{1}{C_3} + \frac{1}{C_4}},$$

$$g_1 = \frac{1}{R_1} = 2\alpha C_3 T_{\max},$$

and

$$g_2 = 2\alpha C_3 \left[\left(1 + \frac{C_4}{C_3} \right) Q^2 - T_{\max} \right].$$

The last equation shows that T_{\max} must be less than or equal to $\left(1 + \frac{C_4}{C_3} \right) Q^2$.

The element g_2 is not necessary to realize the desired poles but it allows the gain to be adjusted.

From Eq. (16), T_{FB} for this circuit is,

$$T_{FB} = \frac{s^2 + 2\alpha s + \omega_o^2}{s^2 + 2\alpha s \left[1 + Q^2 \left(1 + \frac{C_4}{C_3} \right) \right] + \omega_o^2} \quad (29)$$

The minimum magnitude of T_{FB} occurs near $s = i\omega_o$ and it is approximately $Q^{-2} \left(1 + \frac{C_4}{C_3} \right)^{-1}$. So for $|K T_{FB}|$ to be $\gg 1$, K must be much greater than $Q^2 \left(1 + \frac{C_4}{C_3} \right)$, which is like the low-pass circuit in its dependence on Q^2 .

IV. HIGH Q BANDPASS CIRCUITS

In this section two circuits for the realization of a high Q bandpass characteristic are given. The transfer function has a pair of conjugate complex poles and a simple zero at the origin, as in Eq. (25). With the operational amplifiers available commercially today, a stable Q of several hundred is practical. The first circuit uses a Wien bridge arrangement and the second uses a twin-T.

A. Wien Bridge Circuit

The circuit of interest is shown in Fig. 5, and we use the following parameters.

$$Z_1 = R_1 + \frac{1}{sC_1} = \frac{R_1}{s} (s + \sigma_1) , \quad \sigma_1 = 1/R_1C_1$$

$$Y_2 = sC_2 + G_2 = C_2 (s + \sigma_2) , \quad \sigma_2 = G_2/C_2$$

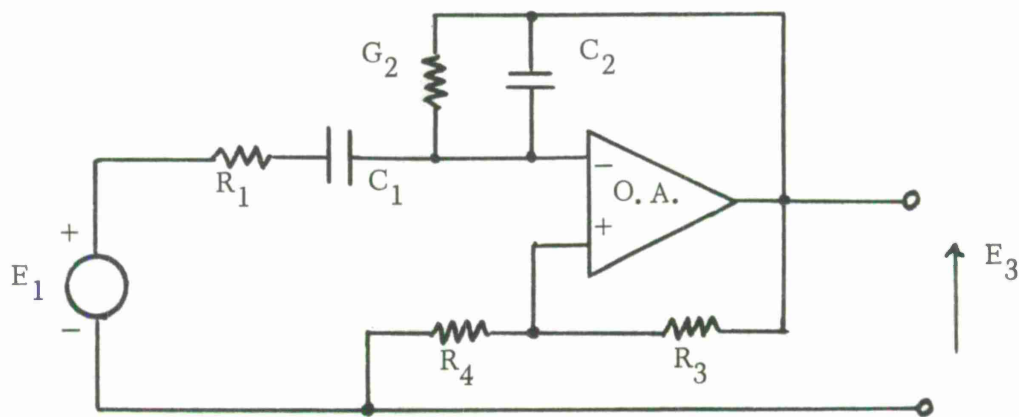
$$\rho = R_3/R_4 \quad \text{and} \quad \gamma = C_1/C_2$$

For the Wien bridge T_F and T_{FB} are

$$T_F = \frac{-1}{1 + Z_1 Y_2}$$

and

$$-T_{FB} = \frac{Z_1 Y_2}{1 + Z_1 Y_2} - \frac{1}{1 + \rho}$$



$$\sigma_1 = \frac{1}{R_1 C_1}$$

$$\rho = \frac{R_3}{R_4}$$

$$\sigma_1 \sigma_2 = \omega_o^2$$

$$\sigma_2 = \frac{1}{R_2 C_2}$$

$$\gamma = \frac{C_1}{C_2}$$

Fig. 5. Wien bridge active bandpass circuit.

So the transfer function of the circuit in Fig. 5 is

$$T(s) = \frac{E_3}{E_1} = \frac{-(1+\rho^{-1})}{Z_1 Y_2 - \rho^{-1}}$$

$$= \frac{-s(1+\rho^{-1})/R_1 C_2}{s^2 + s\left(\sigma_1 + \sigma_2 - \frac{1}{R_1 C_2 \rho}\right) + \sigma_1 \sigma_2} \quad (30)$$

For the design we require $\sigma_1 \sigma_2 = \omega_o^2$ and $2\alpha = \sigma_1 + \sigma_2 - \frac{1}{R_1 C_2 \rho}$. The quantity 2α is equal to the difference of two positive quantities, each of which (to maximize stability) should be as small as possible. Since $\sigma_1 \sigma_2 = \omega_o^2$, $\sigma_1 + \sigma_2$ is minimum when $\sigma_1 = \sigma_2 = \omega_o$ and this result will be used throughout. However this does not require $C_1 = C_2$. We will see from another viewpoint that $C_1 = C_2$ is not optimum. So the transfer function is

$$T(s) = \frac{-s(1+\rho^{-1}) \omega_o \gamma}{s^2 + 2\omega_o s(1 - \frac{\gamma}{2\rho}) + \omega_o^2} \quad (31)$$

and

$$\alpha = \omega_o (1 - \frac{\gamma}{2\rho})$$

or

$$Q = \frac{\omega_o}{2\alpha} = \frac{1}{2 - \gamma/\rho} \quad (32)$$

For high Q $\gamma/\rho \approx 2$.

By looking at T_{FB} we can find a value of ρ that minimizes the sensitivity of the transfer function to changes in amplifier gain. For the Wien bridge circuit T_{FB} is

$$-T_{FB} = \frac{s^2 + 2\omega_o s(1 - \frac{\gamma}{2\rho}) + \omega_o^2}{(1 + \frac{1}{\rho}) [s^2 + \omega_o s(2 + \gamma) + \omega_o^2]}$$

$$= \frac{s^2 + 2\alpha s + \omega_o^2}{(1 + \frac{1}{\rho}) [s^2 + \omega_o s(2 + \gamma) + \omega_o^2]} \quad (33)$$

For $s = i \omega_o$, T_{FB} is very near its minimum value and we get

$$|T_{FB}|_{s=i\omega_o} = \frac{Q^{-1}}{2+\gamma + \frac{2}{\rho} + \frac{\gamma}{\rho}}.$$

For $\gamma/\rho = 2$, the value of $\rho = 1$ minimizes the denominator to a value of 8. So the minimum value of $|T_{FB}|$ as a function of frequency is $(8Q)^{-1}$ which is maximized by the parameters available. This is a very broad optimum, and $\rho = 2$ or $1/2$ decreases the value of $|T_{FB}(i\omega_o)|$ to only $(9Q)^{-1}$. Therefore, for $|K T_{FB}|$ to be $\gg 1$, K must be much greater than $8Q$. This dependence of the minimum value of K on Q (rather than on Q^2 as in the previous circuit) is the reason that the Wien bridge circuit and the following twin-T circuit are preferred for the realization of high- Q poles.

The circuit can be designed easily as follows. Select capacitors and measure their values (for $1/2 \leq \rho \leq 2$, $1 \leq \gamma = C_1/C_2 \leq 4$). Compute the values of R_1 , R_2 and ρ using $\omega_o = 1/R_1 C_1 = 1/R_2 C_2$ and $\rho = (\gamma/2) (1 - \frac{\alpha}{\omega})^{-1}$. Usually for high- Q circuits R_1 and R_4 must be adjusted in the circuit to obtain sufficient accuracy in the realization, so provision for this should be made. R_1 adjusts the center frequency and has a slight effect on Q while R_4 adjusts the Q . The maximum gain occurs at $s = i\omega_o$ and is $|T(i\omega_o)| = (2+\gamma) Q$. If this is too high, the gain can be reduced by using a resistive voltage divider at the input which has the desirable feature of reducing the effect of source impedance variations or uncertainties on the transfer function.

B. Twin-T Circuit

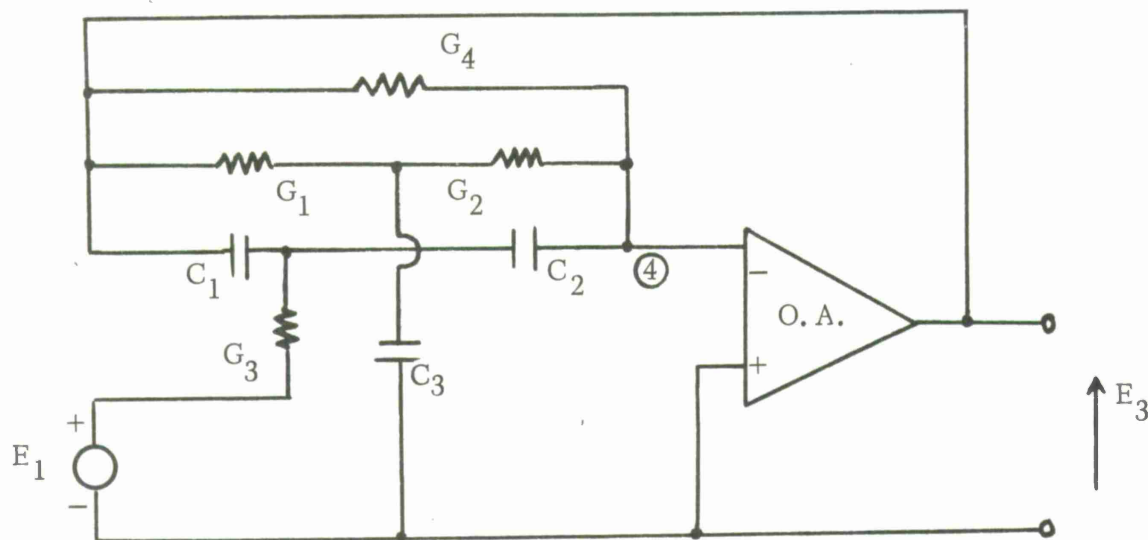
The twin-T circuit in Fig. 6 realizes the same transfer function as the Wien bridge, a pair of complex conjugate poles and a zero at the origin, Eq. (25). Many other twin-T active circuits have a pair of zeros on the negative real axis along with the pair of complex poles. In many cases, zeros at the origin and infinity are desirable.

The analysis of this circuit is straightforward but laborious, so only the results will be given. T_F and T_{FB} are

$$-T_F = \frac{s(G_3/C_1)}{s^2 + s \left(\frac{G_2 + G_4}{C_o} + \frac{G_3}{C_1} \right) + \frac{G_1}{C_o} \left(\frac{G_2}{C_3} + \gamma \sigma_o \right)} \quad (34)$$

$$-T_{FB} = \frac{s^2 + s \left(\frac{G_4}{C_o} \right) + \frac{G_1}{C_o} \left(\frac{G_2}{C_3} + \gamma \sigma_o \right)}{s^2 + s \left(\frac{G_2 + G_4}{C_o} + \frac{G_3}{C_1} \right) + \frac{G_1}{C_o} \left(\frac{G_2}{C_3} + \gamma \sigma_o \right)} \quad (35)$$

In Fig. 6 the parameters are defined in terms of the circuit element values. Thus, the transfer function



$$\sigma_o = \frac{G_3}{C_1 + C_2} = \frac{G_1 + G_2}{C_3} \quad C_o = \frac{C_1 C_2}{C_1 + C_2}$$

$$\gamma = \frac{G_4}{G_1} = \frac{R_1}{R_4}$$

Fig. 6. Twin-T active bandpass circuit.

for very large amplifier gain is

$$T(s) = \frac{E_3}{E_1} = \frac{-s(G_3/C_1)}{s^2 + s \left(\frac{G_4}{C_o} \right) + \frac{G_1}{C_o} \left(\frac{G_2}{C_3} + \gamma \sigma_o \right)} \quad (36)$$

With the parameters as specified in Fig. 6, the twin-T itself (with $G_4 = 0$) has zero transmission at the real frequency $\omega = \sigma_0$. However, even with this balance there are still too many parameters to make an efficient optimization, so the following further restrictions are placed on the circuit element values (these are a standard set of assumptions).

$$\left. \begin{array}{ll} C_1 = C_2 & C_3 = \frac{2}{n} C_1 \\ G_1 = G_2 & G_3 = 2 n G \end{array} \right\} \quad (37)$$

Therefore,

$$C_0 = \frac{1}{2} C_1 \text{ and } \sigma_0 = \frac{G_3}{2C_1} = \frac{2G_1}{C_3} = n \frac{G_1}{C_1} = \frac{1}{n} \frac{G_3}{C_3} .$$

The transfer function now becomes

$$T(s) = \frac{-2n\sigma_1 s}{s^2 + 2\gamma \sigma_1 s + n\sigma_1^2 (1+2\gamma)} \quad (38)$$

where $\sigma_1 = G_1/C_1$, and the feedback factor is now

$$-T_{FB} = \frac{s^2 + 2\gamma \sigma_1 s + n\sigma_1^2 (1+2\gamma)}{s^2 + 2\sigma_1 s (1+n+\gamma) + n\sigma_1^2 (1+2\gamma)} \quad (39)$$

As in the case of the Wien bridge circuit, optimum values of the parameters can be found which minimize the maximum value of the sensitivity to changes in the gain of the operational amplifier. To this end we find the frequency that minimizes $|T_{FB}|$ to $|T_{FB}|_{\min}$, and then find the parameters, n and γ , that maximize $|T_{FB}|_{\min}$. T_{FB} has a pair of negative real poles and a pair of complex zeros which are the poles of the transfer function, Eq. (38). $|T_{FB}|$ has a minimum value which is approximately $|T_{FB}(i\omega_0)|$, where $\omega_0 = \sigma_1 \sqrt{n(1+2\gamma)}$.

$$|T_{FB}(i\omega_0)| = \frac{\gamma}{1+n+\gamma} \quad (40)$$

The parameters n and γ are related through the Q of the pole of the transfer function,

$$Q = \frac{\omega_0}{2\alpha} = \frac{1}{2\gamma} \sqrt{n(1+2\gamma)} \approx \frac{\sqrt{n}}{2\gamma}, \text{ for } \gamma \ll 1.$$

Using this in Eq. (40) we obtain

$$|T_{FB}(i\omega_0)| = \left[1 + 2Q \left(\frac{1+n}{\sqrt{n}} \right) \right]^{-1}$$

which is maximum for $n = 1$ and then has the value $(1+4Q)^{-1}$, and as above the optimum is not critical. So, for this twin-T circuit, the amplifier gain, K , must be much greater than $4Q+1$ to make very small the sensitivity of the transfer function to changes in gain. This is roughly a factor of two improvements over the Wien bridge, but costs an extra capacitor.

The following design procedure is suggested as one that is easy to carry out and it requires the trimming of one capacitor at most (three resistors must be trimmed).

1. Find tentative values for C_1 , C_2 , C_3 and R_1 by carrying out a rough design using Eqs. (37) and (38) for some value of n in the range $1/4 \leq n \leq 4$.
2. Select components for these four elements and measure their values on an impedance bridge.
3. Using Eq. (36) compute the remaining components: G_4 , G_2 and G_3 .

$$G_4 = 2\alpha C_0 = R_4^{-1}$$

$$G_2 = \frac{\omega_0^2 C_0 C_3 R_1 R_4^{-1}}{R_1 + R_4} = R_2^{-1}$$

$$G_3 = \left(\frac{C_1 + C_2}{C_3} \right) (G_1 + G_2) = R_3^{-1}$$

The three elements must be trimmed to their computed values. For a very high Q pole the twin-T must have very high rejection at the frequency $\omega = \sigma_0$, and both R_3 and C_3 must be adjusted in the circuit to obtain this.

V. CIRCUITS WITH ZEROS OF TRANSMISSION

For the realization of a pair of imaginary transmission zeros and a pair of complex poles, the twin-T RC used in the previous section is used, but here it is connected to the operational amplifier in a different way. The connection is basically the same as that of the Wien bridge circuit of Section IV-A. Here the twin-T replaces the RC arms of the Wien bridge. Two similar circuits are given and the general form of the transfer function is

$$\frac{s^2 + \omega_0^2}{s^2 + 2\alpha s + \omega_1^2}$$

Various special forms result in $\omega_1 < \omega_0$, $\omega_1 > \omega_0$ or $\omega_1 = \omega_0$ (which is a notch network).

A. Circuit Alpha

The first circuit for consideration is shown in Fig. 7, and it has $y_{44} = G_5 + G_6$ and $y_{43} = -G_5$. See Eq. (14). Thus, the zeros of y_{21} (which are those of the twin-T) are zeros of transmission. Stability requires that C_4 and G_4 cannot be zero together; in the designs discussed either one or the other is zero.

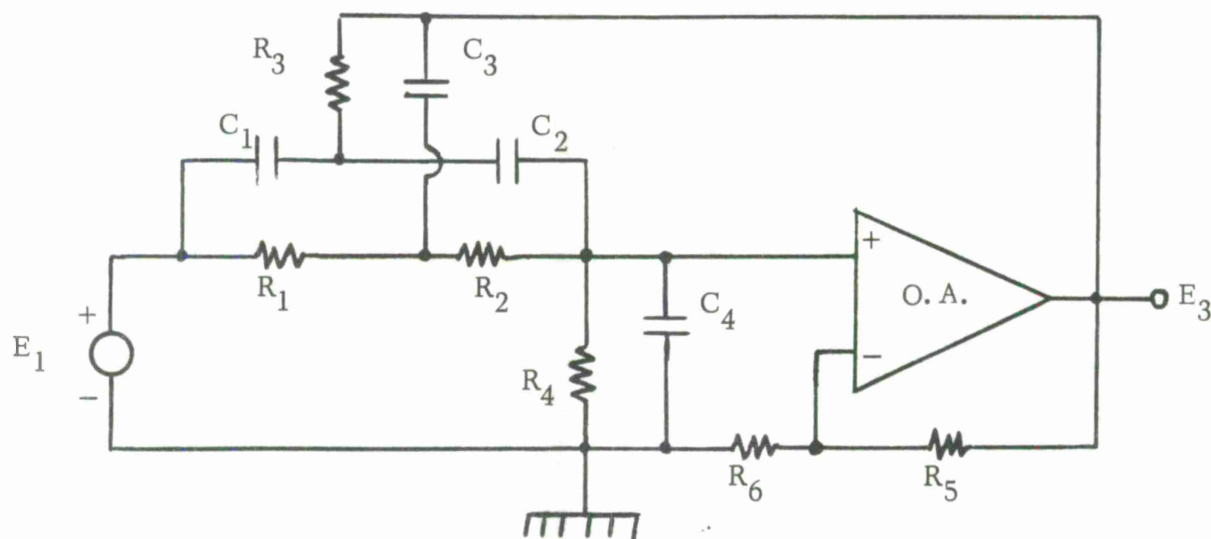


Fig. 7. Active RC twin-T circuit alpha whose transfer function is Eq. (43).

A straightforward but tedious analysis of the circuit yields the following expressions for T_F and T_{FB} .

$$T_F = \frac{-\left(s^2 + \frac{G_1 G_2}{C_o C_3}\right)}{s^2 \left(1 + \frac{C_4}{C_o}\right) + s \left(\frac{\sigma_o C_4 + G_4}{C_o} + \frac{G_3}{C_1} + \frac{G_2}{C_o}\right) + \left(\frac{G_1 G_2}{C_o C_3} + \sigma_o \frac{G_4}{C_o}\right)} \quad (41)$$

$$T_{FB} = \frac{s^2 \left(1 + \frac{C_4}{C_o}\right) + s \left[\frac{\sigma_o C_4 + G_4}{C_o} - \rho \left(\frac{G_3}{C_1} + \frac{G_2}{C_o}\right)\right] + \left(\frac{G_1 G_2}{C_o C_3} + \sigma_o \frac{G_4}{C_o}\right)}{s^2 \left(1 + \frac{C_4}{C_o}\right) + s \left(\frac{\sigma_o C_4 + G_4}{C_o} + \frac{G_3}{C_1} + \frac{G_2}{C_o}\right) + \left(\frac{G_1 G_2}{C_o C_3} + \sigma_o \frac{G_4}{C_o}\right)} \cdot \frac{1}{1+\rho} \quad (42)$$

Thus the transfer function, E_3/E_1 , of the circuit is (for K very large),

$$T(s) = \frac{(1+\rho) \left(s^2 + \frac{G_1 G_2}{C_o C_3}\right)}{s^2 \left(1 + \frac{C_4}{C_o}\right) + s \left[\frac{\sigma_o C_4 + G_4}{C_o} - \rho \left(\frac{G_3}{C_1} + \frac{G_2}{C_o}\right)\right] + \left(\frac{G_1 G_2}{C_o C_3} + \sigma_o \frac{G_4}{C_o}\right)} \quad (43)$$

Here we see that if both C_4 and G_4 are zero, α is negative and $T(s)$ is unstable. However, if only $C_4 = 0$, a stable transfer function with $\omega_1 > \omega_o$ can be obtained; and if only $G_4 = 0$, a stable transfer function with $\omega_1 < \omega_o$ can be obtained. Designs for these two cases will be given, but first optimization of the parameters will be discussed.

As in the previous circuits it is desirable to minimize the sensitivity of the transfer function to changes in amplifier gain K . To accomplish this, the minimum of $|T_{FB}|$ as a function of ω is maximized by a suitable selection of the circuit parameters. In contrast to the previous circuits, a set of parameters to optimize the design of this circuit has not been found. However, it is possible to show that the conditions

$$C_1 = C_2 = \frac{1}{2} C_3 \text{ and } G_1 = G_2 = \frac{1}{2} G_3 \quad (44)$$

nearly satisfy the equation for the optimum design in each of the two cases of interest, and that $|T_{FB}|$ has a minimum value of approximately $(\alpha/2\omega_1) = (4Q)^{-1}$. This is essentially the same as for the circuit in Section IV-B which also uses the twin-T. So this will be used as a satisfactory solution to the optimization of the design.

1. Design for $C_4 = 0$ ($\omega_1 > \omega_0$).

The transfer function is

$$T(s) = \frac{(1+\rho) \left(s^2 + \frac{G_1 G_2}{C_o C_3} \right)}{s^2 + s \left[\frac{G_4}{C_o} - \rho \left(\frac{G_3}{C_1} + \frac{G_2}{C_o} \right) \right] + \left(\frac{G_1 G_2}{C_o C_3} + \sigma_o \frac{G_4}{C_o} \right)} \quad (45)$$

where

$$\omega_o^2 = \frac{G_1 G_2}{C_o C_3} \quad (46)$$

$$\omega_1^2 = \omega_o^2 + \sigma_o \frac{G_4}{C_o} \quad (47)$$

$$2\alpha = \frac{G_4}{C_o} - \rho \left(\frac{G_3}{C_1} + \frac{G_2}{C_o} \right), \quad \rho = \frac{R_5}{R_6} = \frac{G_6}{G_5} \quad (48)$$

$$\sigma_o = \frac{G_3}{C_1 + C_2} = \frac{G_1 + G_2}{C_3} \quad (49)$$

$$C_o = \frac{C_1 C_2}{C_1 + C_2} \quad (50)$$

- a) After selecting a suitable impedance level for the circuit, assume the conditions of Eq. (44) and compute tentative values for the elements of the twin-T using

$$\frac{G_1}{C_1} = \frac{G_3}{C_3} = \omega_o$$

- b) Select components for C_1, C_2, C_3 and G_1 (approximately equal to the computed values) and measure their actual values to an accuracy of $\pm 0.1\%$.

c) From these measured values, compute in succession: G_2 using Eqs. (46) and (50), G_3 and σ_o using Eq. (49), G_4 using Eq. (47), and ρ using Eq. (48).

d) Trim components for G_2 through G_6 to these computed values, again to a tolerance of 0.1% and construct the circuit.

2. Design for $G_4 = 0$ ($\omega_1 < \omega_o$)

The transfer function is

$$T(s) = \frac{(1+\rho) \left(s^2 + \frac{G_1 G_2}{C_o C_3} \right)}{s^2 \left(1 + \frac{C_4}{C_o} \right) + s \left[\sigma_o \frac{C_4}{C_o} - \rho \left(\frac{G_3}{C_1} + \frac{G_2}{C_o} \right) \right] + \frac{G_1 G_2}{C_o C_3}} \quad (51)$$

where ω_o , σ_o and C_o are as above in Eqs. (46), (49) and (50), respectively, and

$$\omega_1^2 = \omega_o^2 \left(1 + \frac{C_4}{C_o} \right)^{-1} \quad (52)$$

$$2\alpha = \left[\sigma_o \frac{C_4}{C_o} - \rho \left(\frac{G_3}{C_1} + \frac{G_2}{C_o} \right) \right] \left(1 + \frac{C_4}{C_o} \right)^{-1}, \quad \rho = \frac{R_5}{R_6} \quad (53)$$

a) Proceed as in parts a) and b) above.

b) From these measured values, compute in succession: G_2 using Eqs. (46) and (50), G_3 and σ_o using Eq. (49), C_4 using Eq. (52), and ρ using Eq. (53).

c) Trim components to these computed values to a tolerance of 0.1% and construct the circuit.

For both designs the tolerance of 0.1% on components is sufficiently accurate so that in most cases no further trimming in the circuit is necessary. In some wide band situations a tolerance of 1% might be sufficient, but in narrow band situations in-circuit trimming of the twin-T notch frequency (ω_o) and of ρ are necessary. C_3 and G_3 provide orthogonal adjustments for ω_o .

B. Circuit Beta

The second circuit for consideration, which is a modified arrangement of the first, is shown in Fig. 8. It is of interest, in particular, because with both C_4 and G_4 equal to zero a notch network with complex poles is obtained. The expressions for T_F and T_{FB} for this circuit are

$$T_F = \frac{-\left(s^2 + \frac{G_1 G_2}{C_o C_3}\right)}{s^2 \left(1 + \frac{C_4}{C_o}\right) + s \left(\frac{\sigma_o C_4 + G_4 + G_2}{C_o} + \frac{G_3}{C_1}\right) + \left(\frac{G_1 G_2}{C_o C_3} + \sigma_o \frac{G_4}{C_o}\right)} \quad (54)$$

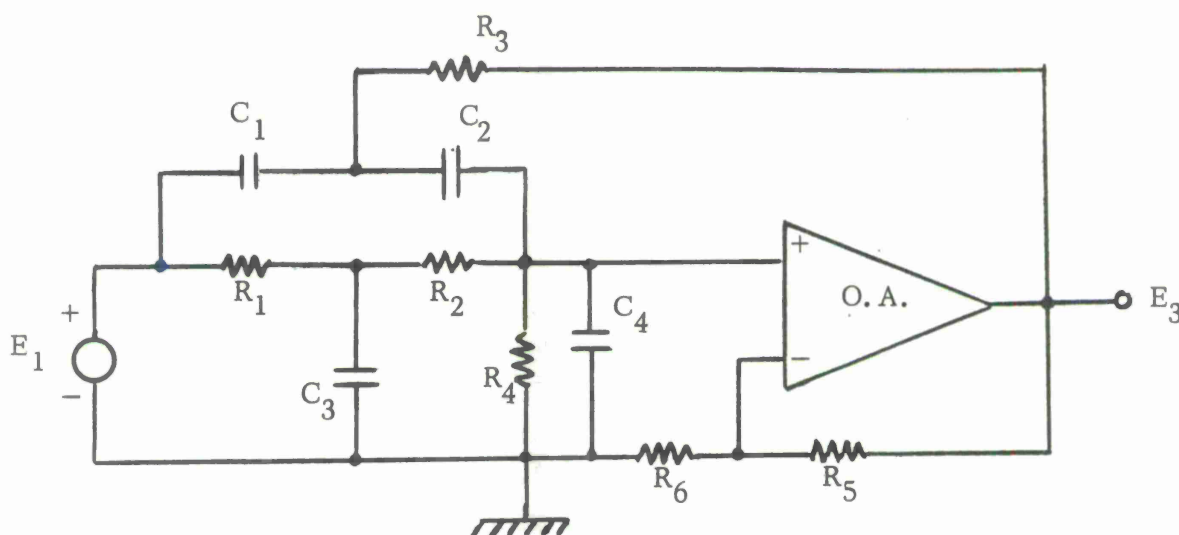


Fig. 8. Active RC twin-T circuit beta whose transfer function is Eq.(56).

and

$$T_{FB} = \frac{s^2 \left(1 + \frac{C_4}{C_o}\right) + s \left(\frac{\sigma_o C_4 + G_4 + G_2}{C_o} - \rho \frac{G_3}{C_1}\right) + \left(\frac{G_1 G_2}{C_o C_3} + \sigma_o \frac{G_4}{C_o}\right)}{s^2 \left(1 + \frac{C_4}{C_o}\right) + s \left(\frac{\sigma_o C_4 + G_4 + G_2}{C_o} + \frac{G_3}{C_1}\right) + \left(\frac{G_1 G_2}{C_o C_3} + \sigma_o \frac{G_4}{C_o}\right)} \cdot \frac{1}{1+\rho} \quad (55)$$

Thus the transfer function of circuit Beta is

$$T(s) = \frac{(1+\rho) \left(s^2 + \frac{G_1 G_2}{C_o C_3} \right)}{s^2 \left(1 + \frac{C_4}{C_o} \right) + s \left(\frac{\sigma_o C_4 + G_4 + G_2}{C_o} - \rho \frac{G_3}{C_1} \right) + \left(\frac{G_1 G_2}{C_o C_3} + \sigma_o \frac{G_4}{C_o} \right)} \quad (56)$$

Minimizing the sensitivity of this circuit has the same difficulties as in circuit Alpha, and fortunately the same near optimum solution is obtained with the conditions of Eq. (44). So the following designs are based on approximate equality in Eq. (44), and the gain of the amplifier is required to be much greater than $4Q$.

1. Design for $C_4 = 0$ ($\omega_1 > \omega_o$)

The transfer function is

$$T(s) = \frac{(1+\rho) \left(s^2 + \frac{G_1 G_2}{C_o C_3} \right)}{s^2 + s \left(\frac{G_2 + G_4}{C_o} - \rho \frac{G_3}{C_1} \right) + \left(\frac{G_1 G_2}{C_o C_3} + \sigma_o \frac{G_4}{C_o} \right)} \quad (57)$$

where ω_o , ω_1 , σ_o and C_o are defined as above in Eqs. (46), (47), (49) and (50) and here

$$2\alpha = \frac{G_2 + G_4}{C_o} - \rho \frac{G_3}{C_1}, \quad \rho = \frac{R_5}{R_6} \quad (58)$$

a) After selecting a suitable impedance level for the circuit, assume the conditions of Eq. (44) and compute tentative values for the elements of the twin-T using

$$\frac{G_1}{C_1} = \frac{G_3}{C_3} = \omega_o$$

b) Select components for C_1 , C_2 , C_3 and G_1 (approximately equal to the computed values) and measure their actual values to an accuracy of 0.1%.

c) From these measured values compute in succession: G_2 using Eqs. (46) and (50), G_3 and σ_o using Eq. (49), G_4 using Eq. (47), and ρ using Eq. (58).

d) Trim components for G_2 through G_6 to these computed values, again to a tolerance of 0.1% and construct the circuit.

2. Design for $G_4 = 0$ ($\omega_1 < \omega_0$)

The transfer function is

$$T(s) = \frac{(1+\rho) \left(s^2 + \frac{G_1 G_2}{C_0 C_3} \right)}{s^2 \left(1 + \frac{C_4}{C_0} \right) + s \left(\frac{\sigma_0 C_4 + G_2}{C_0} - \rho \frac{G_3}{C_1} \right) + \left(\frac{G_1 G_2}{C_0 C_3} \right)} \quad (59)$$

where ω_0 , σ_0 and C_0 have their usual definitions, and here

$$\omega_1^2 = \omega_0^2 \left(1 + \frac{C_4}{C_0} \right)^{-1} \quad (60)$$

and

$$2\alpha = \left(\frac{\sigma_0 C_4 + G_2}{C_0} - \rho \frac{G_3}{C_1} \right) \left(1 + \frac{C_4}{C_0} \right)^{-1}, \quad \rho = \frac{R_5}{R_6} \quad (61)$$

a) Proceed as in a) and b) immediately above.

b) From the measured values for C_1, C_2, C_3 and G_1 compute in succession: G_2 using Eqs. (46) and (50), G_3 and σ_0 using Eq. (49), C_4 using Eq. (60), and ρ using Eq. (61).

c) Trim components to these computed values to a tolerance of 0.1% and construct the circuit.

3. Design for $C_4 = G_4 = 0$ ($\omega_1 = \omega_0$)

The transfer function is

$$T(s) = \frac{(1+\rho) \left(s^2 + \frac{G_1 G_2}{C_0 C_3} \right)}{s^2 + s \left(\frac{G_2}{C_0} - \rho \frac{G_3}{C_1} \right) + \left(\frac{G_1 G_2}{C_0 C_3} \right)} \quad (62)$$

which has the complex poles at the same distance from the origin as the transmission zeros. The magnitude of Eq. (62) is constant except for a narrow notch centered at ω_0 and is 3 dB down at $\omega_0 \pm \alpha$. In Eq. (62) ω_0 , σ_0 and C_0 have their usual definitions, and $\omega_1 = \omega_0$ while,

$$2\alpha = \frac{G_2}{C_0} - \rho \frac{G_3}{C_1}, \quad \rho = \frac{R_5}{R_6} \quad (63)$$

- a) Proceed as in a) and b) above for the case $C_4 = 0$.
- b) From the measured values of C_1, C_2, C_3 and G_1 , compute in succession: G_2 using Eqs. (46) and (50), and ρ using Eq. (63).
- c) Trim components to these computed values to a tolerance of 0.1% and construct the circuit.

For accurate placement of the notch frequency, provision must be made to trim G_3 and C_3 in the circuit.

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